Model Order Reduction of the Navier-Stokes Equations at High Reynolds Number

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Common elements in most Model Order Reduction (MOR) techniques:

1. Spectral discretization

$$\mathbf{u}(\mathbf{x},t) \approx \sum_{i=1}^{n} a_i(t)\mathbf{u}_i(\mathbf{x}) \tag{1}$$

2. Projection

$$\left\langle \mathbf{v}_i, R\left(\sum_{i=1}^{n} a_i \mathbf{u}_i\right)\right\rangle_{\Omega} = 0 \tag{2}$$

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- One popular approach: first n left-singular vectors of M, i.e $\mathbf{u}_i = U(:, i)$ where $M = U\Sigma V^T$ and

$$M = \begin{bmatrix} \mathbf{u}(\mathbf{x}_1, t_1) & \mathbf{u}(\mathbf{x}_1, t_2) & \cdots & \mathbf{u}(\mathbf{x}_1, t_{N_t}) \\ \mathbf{u}(\mathbf{x}_2, t_1) & \mathbf{u}(\mathbf{x}_2, t_2) & \cdots & \mathbf{u}(\mathbf{x}_2, t_{N_t}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{u}(\mathbf{x}_{N_x}, t_1) & \mathbf{u}(\mathbf{x}_{N_x}, t_2) & \cdots & \mathbf{u}(\mathbf{x}_{N_x}, t_{N_t}) \end{bmatrix} \quad (3)$$

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- Turbulence is a multi-scale phenomenon: large scale flow features are broken down into smaller and smaller scales until the scales are fine enough that viscous forces can dissipate their energy.
- Application of any POD-based MOR strategy to a turbulent flow is problematic because POD, by construction, is biased toward the large, energy containing scales of the turbulent flow.
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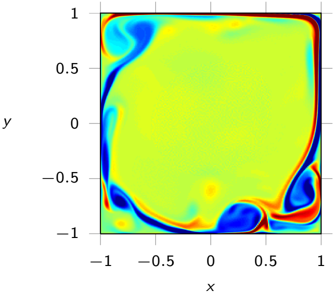
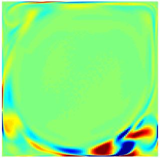
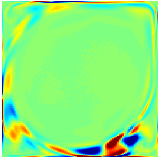
Background and motivation

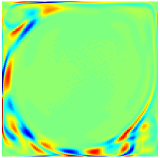
Classical benchmark: Incompressible flow inside a square, two-dimensional lid-driven cavity at $\mathrm{Re_u} = 3 \times 10^4$

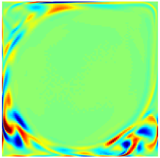


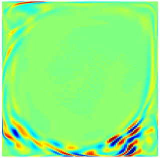
POD basis functions of the lid-driven cavity

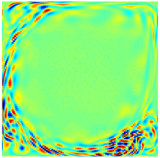

(a) $\mathbf{u}_1$


(b) $\mathbf{u}_2$


(c) $\mathbf{u}_{10}$


(d) $\mathbf{u}_{20}$


(e) $\mathbf{u}_{50}$


(f) $\mathbf{u}_{200}$

Background and motivation

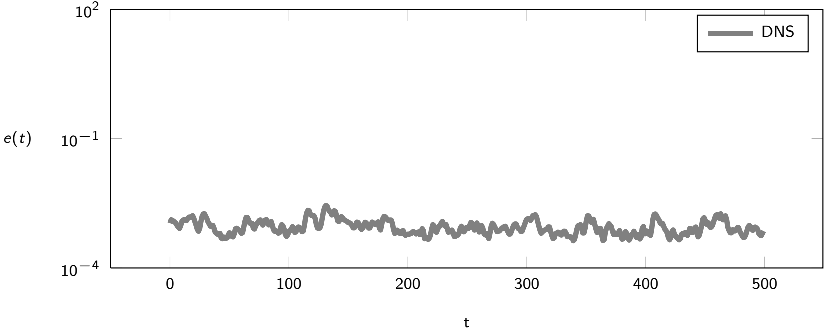
Percent of turbulent kinetic energy, $e(t) \equiv 1/2 \int_\Omega |\mathbf{u}(\mathbf{x}, t)|^2 \, d\mathbf{x}$ captured by the first n basis functions, $\mathbf{u}_i$ of the lid-driven cavity

n	%
1	16.06
2	29.21
3	37.45
4	44.88
5	50.37
10	67.16
20	82.40
50	93.21
200	99.31

Background and motivation

Standard POD-Galerkin ROMs of the lid-driven cavity

- ROM performance quantified using the turbulent kinetic energy, $e(t) \equiv 1/2 \int_\Omega |\mathbf{u}(\mathbf{x}, t)|^2 \, d\mathbf{x}$



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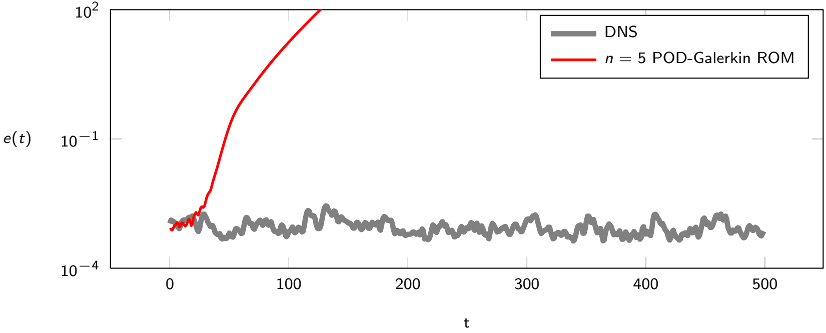

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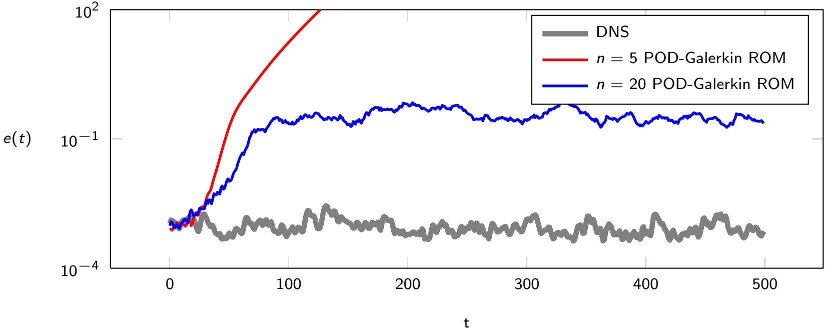

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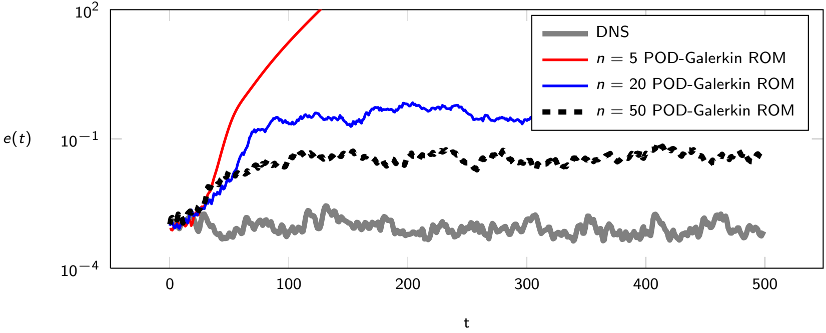

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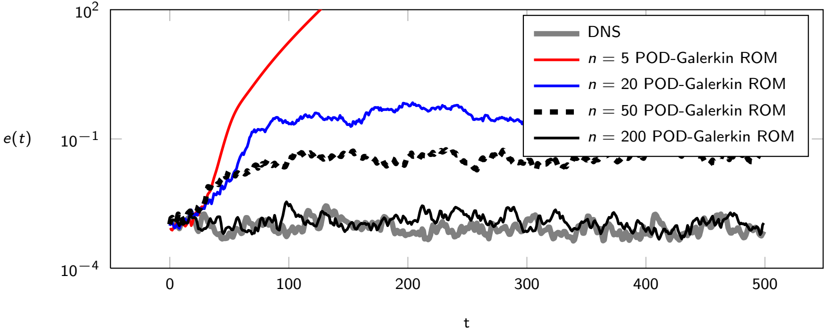

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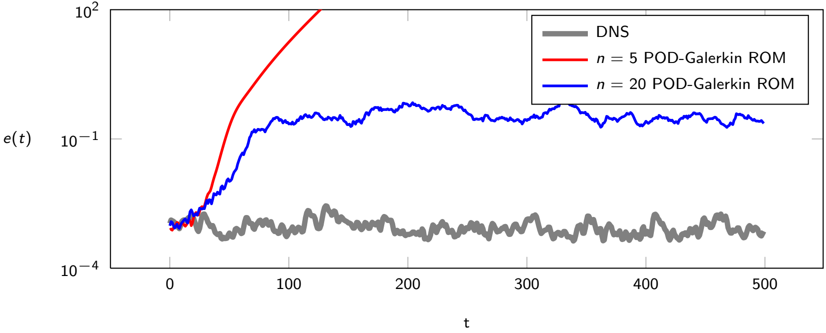

Proposed new approach

Basic idea: Instead of using *just* the most energetic POD basis functions, include a few lower energy POD basis functions so that dissipative scales are resolved.

- For example, consider we are interested in forming a $n = 3$ ROM.
- Standard approach: use the first 3 most energetic POD basis functions, i.e. $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$
- Proposed Approach: $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_5$

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Generalization: We search for the n new basis functions in the range of N standard POD basis functions (where $N > n$).

- If we label the n new basis functions as $\tilde{\mathbf{u}}_i$
- We can write

$$\tilde{U} = UX \tag{4}$$

where $X \in \mathbb{R}^{N\times n}$ is an orthonormal ($X^T X = I_{n\times n}$) transformation matrix and the basis functions are vectorized and assembled as follows

$$U = \begin{bmatrix} \operatorname{vec}(\mathbf{u}_1) & \operatorname{vec}(\mathbf{u}_2) & \cdots & \operatorname{vec}(\mathbf{u}_N) \end{bmatrix} \tag{5a}$$

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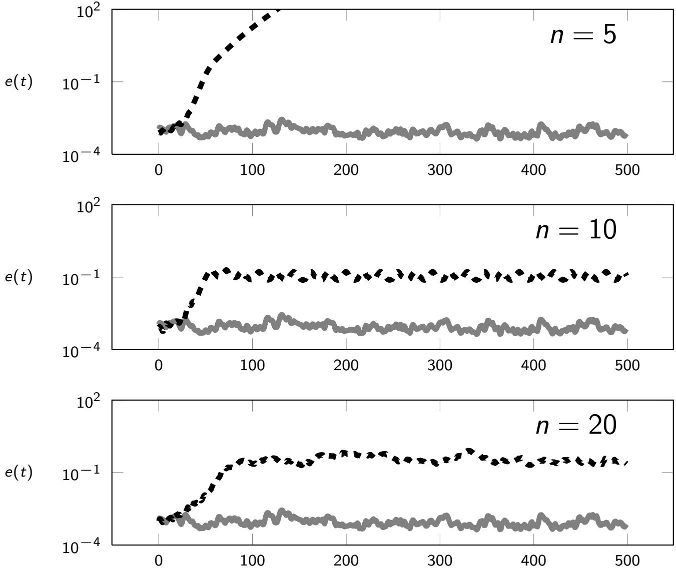
We have developed an algorithm for X that uses the turbulent kinetic energy (tke) equation as a side constraint to the standard POD/SVD snapshot approach. The algorithm is computationally efficient and thus can be implemented using standard MATLAB constrained optimization algorithms. Currently, our algorithm is limited to steady Dirichlet, ambient flow or free-stream conditions at infinity, or periodic boundary conditions.

[1] Balajewicz, M., Dowell, E., & Noack, B. 2012, "A Novel Model Order Reduction Approach for Navier-Stokes Equations at High Reynolds Number", arXiv:1211.1720, (Under consideration for publication in J. Fluid Mech.)

Lid driven cavity, revisited



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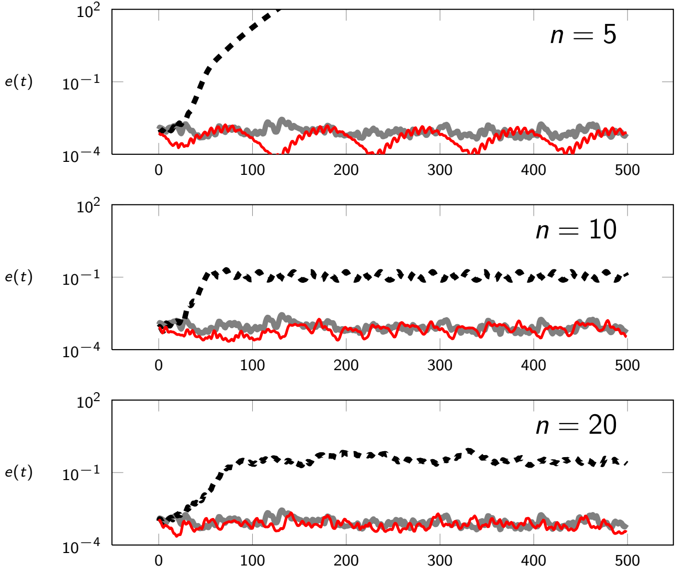

Conclusions

- Our approach models small scales of the flow *directly* using basis functions that are different from the standard POD basis functions.
- For many boundary conditions, a computationally efficient algorithm that can be implemented in MATLAB using `fmincon` is available.
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